M. Spinks
R. Veroff

Constructive Logic with Strong Negation is a Substructural Logic. II

Abstract. The goal of this two-part series of papers is to show that constructive logic with strong negation $\mathbf{N}$ is definitionally equivalent to a certain axiomatic extension $\mathbf{NFL}_{ew}$ of the substructural logic $\mathbf{FL}_{ew}$. The main result of Part I of this series [41] shows that the equivalent variety semantics of $\mathbf{N}$ (namely, the variety of Nelson algebras) and the equivalent variety semantics of $\mathbf{NFL}_{ew}$ (namely, a certain variety of FL_{ew}-algebras) are term equivalent. In this paper, the term equivalence result of Part I [41] is lifted to the setting of deductive systems to establish the definitional equivalence of the logics $\mathbf{N}$ and $\mathbf{NFL}_{ew}$. It follows from the definitional equivalence of these systems that constructive logic with strong negation is a substructural logic.

Keywords: Constructive logic, strong negation, substructural logic, Nelson algebra, $\mathcal{FL}_{ew}$-algebra, residuated lattice.

1. Introduction

Let $\Sigma[\mathbf{IPC}]$ denote the Hilbert-style presentation of Blok and Pigozzi [6, Example 2.2.2] of the intuitionistic propositional calculus $\mathbf{IPC}$ over the language type $\Lambda[\mathbf{IPC}] := \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$, where $\wedge, \vee, \rightarrow$ are binary logical connectives, $\neg$ is a unary logical connective, and $\mathbf{0}$ and $\mathbf{1}$ are nullary logical connectives respectively. *Constructive logic with strong negation*, denoted $\mathbf{N}$, is the deductive system over the language type $\Lambda[\mathbf{N}] := \Lambda[\mathbf{IPC}] \cup \{\sim\}$, where $\sim$ is a unary logical connective (called the *strong negation*), determined by the axioms and inference rules of $\Sigma[\mathbf{IPC}]$ together with the axioms [42]:

$$\begin{array}{cc}
\sim p \rightarrow (p \rightarrow q) & \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q) \\
\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q) & \sim(\neg p) \leftrightarrow p \\
\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q) & \sim(\sim p) \leftrightarrow p.
\end{array}$$

(Here we are abbreviating $(p \rightarrow q) \wedge (q \rightarrow p)$ by $p \leftrightarrow q$.) By [34, Chapter XII], $\mathbf{N}$ is strongly and regularly algebraisable in the sense of [15]. The study of constructive logic with strong negation has been pursued extensively in the literature [34, 42, 37]; for a brief discussion and overview, see Wójcicki [47, Section 5.3.0].

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Let **FL** denote the sequent system of Galatos *et al.* [19, Section 2.1.3], over the language $\Lambda[\mathbf{FL}] := \langle \wedge, \vee, *, \backslash, /, \mathbf{0}, \mathbf{1} \rangle$ of type $\langle 2,2,2,2,2,0,0 \rangle$, obtained from the Gentzen sequent calculus **LJ** by deleting all the structural rules together with the logical rules for implication, and then adding rules for the division connectives $\backslash$ and $/$ and the fusion connective $*$.[1] The *full Lambek calculus*, also denoted **FL**, is the deductive system determined by the sequent system **FL** in the sense that for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\Lambda[\mathbf{FL}]}$, $\Gamma \vdash_{\mathbf{FL}} \varphi$ if and only if $\{(\rhd \psi) : \psi \in \Gamma\} \vdash^{\mathbf{FL}} (\rhd \varphi)$. (Here $S' \vdash^{\mathbf{FL}} s$ if there is a proof in **FL** of the sequent s from the set of sequents S', while the auxiliary symbol $\rhd$ denotes the separator of an arbitrary sequent $\psi_1, \ldots, \psi_n \rhd \varphi$.) By [18, Theorem 3.2], **FL** is strongly algebraisable in the sense of [15]. For studies of **FL**, see [30, 18, 19].

Let (e), (c), (i), and (o) denote the structural rules of exchange, contraction, left weakening, and right weakening respectively, as given in [19, Section 2.1.1]. For $S \subseteq \{e, c, i, o\}$, let $\mathbf{FL}_S$ denote the extension of **FL** obtained by adjoining the structural rules $\{(s) : s \in S\}$ to **FL**. (Following the practice of [19], we abbreviate the combination $\{i, o\} \subseteq S$ by w.) Recall that, in the presence of the exchange rule, the formulas $\varphi \backslash \psi$ and ψ / φ are provably equivalent (in the sense of [19, Section 2.1.2]) for all $\varphi, \psi \in \mathrm{Fm}_{\Lambda[\mathbf{FL}]}$ [19, Lemma 2.3]. When $e \in S$, therefore, we fix the language type of $\mathbf{FL}_S$ as $\{\wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1}\}$, where $\Rightarrow$ is a binary logical connective. Thus the *full Lambek calculus with exchange and weakening*, in symbols $\mathbf{FL}_{ew}$, is the deductive system over the language $\Lambda[\mathbf{FL}_{ew}] := \langle \wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ of type $\langle 2,2,2,2,0,0 \rangle$ determined by the sequent system $\mathbf{FL}_{ew}$ $(= \mathbf{FL}_{eio})$.[2] By [18, Theorem 3.3, Theorem 3.4] $\mathbf{FL}_{ew}$ is strongly and regularly algebraisable in the sense of [15]. For studies of $\mathbf{FL}_{ew}$, see in particular [28, 23, 29, 30, 18, 19].

The aim of this two-part series of papers is to show that constructive logic with strong negation is definitionally equivalent to the axiomatic extension $\mathbf{NFL}_{ew}$ of the deductive system $\mathbf{FL}_{ew}$ by the axioms

$$\sim\sim p \Rightarrow p \qquad \text{(Double Negation)}$$

$$\big(p \wedge (q \vee r)\big) \Rightarrow \big((p \wedge q) \vee (p \wedge r)\big) \qquad \text{(Distributivity)}$$

$$\big(p \Rightarrow (p \Rightarrow (p \Rightarrow q))\big) \Rightarrow \big(p \Rightarrow (p \Rightarrow q)\big) \qquad \text{(3-potency)}$$

$$\big((p \Rightarrow (p \Rightarrow q)) \wedge (\sim q \Rightarrow (\sim q \Rightarrow \sim p))\big) \Rightarrow (p \Rightarrow q) \qquad \text{(Nelson).}$$

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[1] Following Girard [20], throughout this paper the structural rules comprise the exchange, (left, right) weakening, and contraction rules. In particular, neither identity nor cut count as a structural rule.

[2] For a sequent system for $\mathbf{FL}_{ew}$ over the language type $\Lambda[\mathbf{FL}_{ew}]$ see Kowalski and Ono [23, Section 1, p. 9].

(Here we are abbreviating $p \Rightarrow \mathbf{0}$ by $\sim p$.)

The proof of this result is in two parts, with one part per paper. In Part I of this series [41] it was shown that the equivalent variety semantics of $\mathbf{N}$ (namely, the variety $\mathcal{N}$ of Nelson algebras [34, Chapter V]) and the equivalent variety semantics of $\mathbf{NFL}_{ew}$ (namely, a certain variety $\mathcal{NFL}_{ew}$ of FL_{ew}-algebras) are term equivalent. For a précis of Part I [41], see Section 2.2 below. In this paper, we lift the term equivalence result of Part I [41] to the setting of deductive systems to establish the definitional equivalence of the logics $\mathbf{N}$ and $\mathbf{NFL}_{ew}$. From the definitional equivalence of these systems we obtain the desired corollary that constructive logic with strong negation is a substructural logic.

The main result of this paper is

THEOREM 1.1.

1. *The map* $\delta : \Lambda[\mathbf{FL}_{ew}] \to \mathrm{Fm}_{\Lambda[\mathbf{N}]}$ *defined by*

$$
\begin{aligned}
p \wedge q &\mapsto p \wedge q \\
p \vee q &\mapsto p \vee q \\
p * q &\mapsto \sim(p \to \sim q) \vee \sim(q \to \sim p) && (*_{\mathrm{def}}) \\
p \Rightarrow q &\mapsto (p \to q) \wedge (\sim q \to \sim p) && (\Rightarrow_{\mathrm{def}}) \\
\mathbf{0} &\mapsto \mathbf{0} \\
\mathbf{1} &\mapsto \mathbf{1}
\end{aligned}
$$

   is an interpretation of $\mathbf{NFL}_{ew}$ *in* $\mathbf{N}$.

2. *The map* $\varepsilon : \Lambda[\mathbf{N}] \to \mathrm{Fm}_{\Lambda[\mathbf{FL}_{ew}]}$ *defined by*

$$
\begin{aligned}
p \wedge q &\mapsto p \wedge q \\
p \vee q &\mapsto p \vee q \\
p \to q &\mapsto p \Rightarrow (p \Rightarrow q) && (\to_{\mathrm{def}}) \\
\neg p &\mapsto p \Rightarrow (p \Rightarrow \mathbf{0}) && (\neg_{\mathrm{def}}) \\
\sim p &\mapsto p \Rightarrow \mathbf{0} && (\sim_{\mathrm{def}}) \\
\mathbf{0} &\mapsto \mathbf{0} \\
\mathbf{1} &\mapsto \mathbf{1}
\end{aligned}
$$

   is an interpretation of $\mathbf{N}$ *in* $\mathbf{NFL}_{ew}$.

3. *The interpretations* δ *and* ε *are mutually inverse.*

Hence the deductive systems $\mathbf{N}$ *and* $\mathbf{NFL}_{ew}$ *are definitionally equivalent.*

A deductive system $\mathbf{S}$ over a language type Λ is said to be *Fregean* if the relativised interderivability relation $\dashv\vdash^T_{\mathbf{S}}$ (T a theory of $\mathbf{S}$) is a congruence relation on the formula algebra $\mathbf{Fm}_\Lambda$. A logic $\mathbf{S}$ is said to be *non-Fregean* if it is not Fregean. A *substructural logic over* $\mathbf{FL}_S$, $S \subseteq \{e, c, i, o\}$, is a deductive system $\mathbf{S}$ that is definitionally equivalent to a non-Fregean extension of $\mathbf{FL}_S$. For a justification of this definition, see Section 3 below.

The main result of this series of papers is

THEOREM 1.2. *Constructive logic with strong negation is a substructural logic over* $\mathbf{FL}_{ew}$.

The following example illustrates Theorems 1.1 and 1.2.

EXAMPLE 1.3. *Classical constructive logic with strong negation*, in symbols $\mathbf{N}_c$, is the axiomatic extension of $\mathbf{N}$ by the Peirce law $\big((p \to q) \to p\big) \to p$. Let $N_c := \{0, a, 1\}$ and consider the operations $\wedge, \vee, \to, \neg$, and $\sim$ defined on N_c by means of the following tables:

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$\to$	0	a	1
0	1	1	1
a	1	1	1
1	0	a	1

$\neg$	
0	1
a	1
1	0

$\sim$	
0	1
a	a
1	0

By Rasiowa [34, Chapter V§3] the algebra $\mathbf{N}_c := \langle N_c; \wedge, \vee, \to, \neg, \sim, 0, 1\rangle$ is, to within isomorphism, the unique 3-element Nelson algebra, and by a well known observation of Vakarelov [42, Theorem 10], $\mathbf{N}_c$ is the deductive system determined by the logical matrix $\big\langle \mathbf{N}_c; \{\mathbf{1}^{\mathbf{N}_c}\}\big\rangle$.

Let $\mathbf{N}_c^\delta$ denote the $\{\wedge, \vee, *, \Rightarrow, 0, 1\}$-term reduct of $\mathbf{N}_c$, where δ is the map of Theorem 1.1(1) above (more precisely, of Theorem 2.1(1) below). It is readily verified that the operations of $\mathbf{N}_c^\delta$ have tables:

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$*$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

$\Rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

From direct inspection of these tables, it is easy to see that $\mathbf{N}_c^\delta$ is term equivalent to the unique (to within isomorphism) 3-element Wajsberg algebra $\mathbf{WA}_2 := \big\langle \{0, a, 1\}; \Rightarrow, \sim, 1\big\rangle$. (For information about Wajsberg algebras, see [5, Section 1, pp. 562–564].) It follows that $\mathbf{N}_c$ is definitionally equivalent to the deductive system determined by the logical matrix $\langle \mathbf{WA}_2; \{\mathbf{1}^{\mathbf{WA}_2}\}\rangle$, *viz.*, the three-valued logic $\mathbf{Ł}_3$ of Łukasiewicz [24].[3] This explains the well known result of Vakarelov [42, Theorem 11] asserting that the axiomatic expansion of classical propositional logic by strong negation is definitionally equivalent to $\mathbf{Ł}_3$.[4] ■

The remainder of this paper is devoted to establishing Theorem 1.1. After attending to numerous preliminaries in Section 2, we give necessary and sufficient conditions in Section 3 for a deductive system to be a substructural logic over $\mathbf{FL}_S$ (in the sense of this paper). Section 4 is devoted to establishing a sufficient condition for two regularly algebraisable deductive systems to be definitionally equivalent. This condition allows us to lift the term equivalence result of Part I [41] directly to the setting of deductive systems in this paper. In Section 5 we present a Hilbert-style axiomatisation of $\mathbf{NFL}_{ew}$ and combine the technical results of Section 4 with the main result of Part I [41] to conclude that the deductive systems $\mathbf{N}$ and $\mathbf{NFL}_{ew}$ are definitionally equivalent. From the definitional equivalence of $\mathbf{N}$ and $\mathbf{NFL}_{ew}$, we finally obtain the desired corollary that constructive logic with strong negation is a substructural logic.

All the proofs of Part I of this series [41], together with the proofs of two lemmas of this paper (Lemmas 5.1 and 5.5), were obtained with the assistance of the automated reasoning program Prover9 [26], using the method of proof sketches [46]. Prover9 is a resolution-based theorem prover for first-order logic with equality that has been shown to be particularly useful in the investigation of (quasi-) equational theories where standard semantic methods cannot readily be applied. For examples of the application of automated reasoning to a wide range of problems in equational logic, see in particular [25].

For the sake of completeness, the automated proofs for Lemmas 5.1 and 5.5 of this paper are included in Appendix A. The website accompanying this series [40] contains the full set of automated proofs supporting both this work and Part I of this series [41].

2. Preliminaries

In this section we fix some terminology and notation that will be used throughout this paper (Section 2.1); recapitulate the main result of Part I of

[3] By Blok and Pigozzi [5, Corollary 3.9], the variety generated by $\mathbf{WA}_2$ is a discriminator variety. Hence, this example also clarifies the characterisation of discriminator varieties of Nelson algebras given in [38, Corollary 5.3].

[4] This situation is called 'strange' by Vakarelov in [43, Section 1, p. 394].

this series [41] (Section 2.2); describe the notion of definitional equivalence exploited in this paper (Sections 2.3–2.4); and summarise some elements of the theory of regularly algebraisable logics (Sections 2.5–2.6).

2.1. Terminology and notation

We adhere to the terminology and notation introduced in Part I of this series [41]. In particular, $\mathbf{X} := \{v_i : i \in \omega\}$ is a countably infinite set of *variables*. Generally we find it convenient to write p, q, r [resp. x, y, z] etc., possibly with subscripts, as metavariables ranging over $\mathbf{X}$ in a logical [resp. algebraic] context. As in Part I [41], for typographical convenience we often denote the application of the function f to a by a^f. Given a set A, $\wp(A)$ denotes the power set of A.

Let Λ be a language type. A Λ-*formula*, or *formula* for short, is an element of the universe $\mathrm{Fm}_\Lambda(\mathbf{X})$ of the absolutely free algebra $\mathbf{Fm}_\Lambda(\mathbf{X})$ of type Λ generated by $\mathbf{X}$. Occasionally we write formulas using Polish prefix notation. We identify the n-ary logical connective $c \in \Lambda$ with the formula $c^{\mathbf{Fm}_\Lambda}(v_0, \ldots, v_{n-1})$ [21, Section 1.1.3, p. 8]. A Λ-*substitution*, or more briefly *substitution*, is an endomorphism of the formula algebra $\mathbf{Fm}_\Lambda(\mathbf{X})$. By the freeness of $\mathbf{Fm}_\Lambda(\mathbf{X})$, we identify any substitution with its restriction to $\mathbf{X}$.

Let $\mathcal{K}$ be a quasivariety and let $\mathbf{A} \in \mathcal{K}$. A $\mathcal{K}$-*congruence* on $\mathbf{A}$ is any congruence θ on $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathcal{K}$. The set of all $\mathcal{K}$-congruences on $\mathbf{A}$ is denoted $\mathrm{Con}_\mathcal{K}\,\mathbf{A}$. For $a, b \in A$, $\Theta^{\mathbf{A}}_\mathcal{K}(a, b)$ denotes the principal $\mathcal{K}$-congruence on $\mathbf{A}$ generated by a, b. We drop all instances of the subscript when $\mathcal{K}$ is a variety.

A *constant term* of a quasivariety $\mathcal{K}$ is a term $t(x_0, \ldots, x_{n-1})$ in the language of $\mathcal{K}$ having the property that $\mathcal{K} \models t(x_0, \ldots, x_{n-1}) \approx t(y_0, \ldots, y_{n-1})$, where the $y_0, \ldots, y_{n-1}$ are new variables distinct from $x_0, \ldots, x_{n-1}$. $\mathcal{K}$ is said to be *pointed* if it has a constant term. By [15, Section 1.5, p. 39] every pointed quasivariety is term equivalent to a quasivariety over a language type with a distinguished constant (*i.e.*, nullary operation) symbol $\mathbf{1}$. In the sequel we always distinguish a constant term in every pointed quasivariety and assume that $\mathbf{1}$ denotes this distinguished constant term.

2.2. Nelson algebras and Nelson $\mathbf{FL}_{ew}$-algebras

A *Nelson algebra* is an algebra $\langle A; \wedge, \vee, \rightarrow, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 1, 0, 0\rangle$ where $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra [2, Chapter XI] and the following identities are satisfied [10]:

$$(x \wedge {\sim} x) \wedge (y \vee {\sim} y) \approx x \wedge {\sim} x \tag{N1}$$

$$x \to x \approx \mathbf{1} \tag{N2}$$

$$(x \to y) \wedge ({\sim} x \vee y) \approx {\sim} x \vee y \tag{N3}$$

$$x \wedge ({\sim} x \vee y) \approx x \wedge (x \to y) \tag{N4}$$

$$(x \to y) \wedge (x \to z) \approx x \to (y \wedge z) \tag{N5}$$

$$(x \wedge y) \to z \approx x \to (y \to z) \tag{N6}$$

$$\neg x \approx x \to \mathbf{0}. \tag{N7}$$

Clearly the class $\mathcal{N}$ of all Nelson algebras is equationally definable. Informally, a Nelson algebra may be understood as a De Morgan algebra $\langle A; \wedge, \vee, {\sim}, 0, 1\rangle$ structurally enriched with a certain weak implication operation $\to$ generalising relative pseudocomplementation [13, Section 3]. For studies of Nelson algebras, see [34, 42, 37, 13].

A *residuated lattice* is an algebra $\langle A; \wedge, \vee, *, \backslash, /, 1\rangle$ of type $\langle 2, 2, 2, 2, 2, 0\rangle$ where $\langle A; \wedge, \vee\rangle$ is a lattice (with lattice ordering $\leq$), $\langle A; *, 1\rangle$ is a monoid, and the equivalences $a * b \leq c$ if and only if $b \leq a\backslash c$ if and only if $a \leq c/b$ are identically satisfied. A residuated lattice $\mathbf{A}$ is said to be *commutative* if its satisfies the identity $x * y \approx y * x$, *contractive* if $a \leq a * a$ for all $a \in A$, and *integral* if $a \leq 1$ for all $a \in A$. By [8, Proposition 4.1] the class of residuated lattices is a variety.

An *FL-algebra* $\langle A; \wedge, \vee, *, \backslash, /, 0, 1\rangle$ is a residuated lattice with distinguished element $0 \in A$. It is easy to see an FL-algebra is commutative if and only if it satisfies the identity $x/y \approx y\backslash x$ [18, Section 2, p. 282]. For this reason we fix the language type of the variety of commutative FL-algebras (and its subvarieties) as $\{\wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1}\}$, where $\Rightarrow$ is a binary operation symbol. Thus an *FL$_{eci}$-algebra* $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a commutative, contractive, integral residuated lattice with distinguished element $0 \in A$. An *FL$_{ew}$-algebra* $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a commutative, integral residuated lattice with distinguished element $0 \in A$ where $0 \leq a$ for all $a \in A$. For studies of FL$_{ew}$-algebras, see [28, 23, 29, 30].

A *Nelson FL$_{ew}$-algebra* is an FL$_{ew}$-algebra satisfying the identities:

$${\sim}{\sim} x \approx x \tag{DN}$$

$$(x \vee y) \wedge (x \vee z) \approx x \vee (y \wedge z) \tag{D7}$$

$$(x \wedge y) \vee (x \wedge z) \approx x \wedge (y \vee z) \tag{D8}$$

$$x \Rightarrow \big(x \Rightarrow (x \Rightarrow y)\big) \approx x \Rightarrow (x \Rightarrow y) \tag{E$_2$}$$

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \approx x \Rightarrow y \tag{N}$$

where ${\sim} x$ abbreviates the term $x \Rightarrow \mathbf{0}$. By [41, Section 2.4] the class $\mathcal{NFL}_{ew}$ of all Nelson FL$_{ew}$-algebras is a variety.

The main result of Part I of this series [41] states

THEOREM 2.1. [41, Theorem 1.1]

1. *The map* $\delta : \Lambda[\mathbf{FL}_{ew}] \to \mathrm{Fm}_{\Lambda[\mathbf{N}]}$ *defined by*

$$x \wedge y \mapsto x \wedge y$$
$$x \vee y \mapsto x \vee y$$
$$x * y \mapsto {\sim}(x \to {\sim} y) \vee {\sim}(y \to {\sim} x) \qquad (*_{\mathrm{def}})$$
$$x \Rightarrow y \mapsto (x \to y) \wedge ({\sim} y \to {\sim} x) \qquad (\Rightarrow_{\mathrm{def}})$$
$$\mathbf{0} \mapsto \mathbf{0}$$
$$\mathbf{1} \mapsto \mathbf{1}$$

is an interpretation of $\mathcal{NFL}_{ew}$ *in* $\mathcal{N}$.

2. *The map* $\varepsilon : \Lambda[\mathbf{N}] \to \mathrm{Fm}_{\Lambda[\mathbf{FL}_{ew}]}$ *defined by*

$$x \wedge y \mapsto x \wedge y$$
$$x \vee y \mapsto x \vee y$$
$$x \to y \mapsto x \Rightarrow (x \Rightarrow y) \qquad (\to_{\mathrm{def}})$$
$$\neg x \mapsto x \Rightarrow (x \Rightarrow \mathbf{0}) \qquad (\neg_{\mathrm{def}})$$
$${\sim} x \mapsto x \Rightarrow \mathbf{0} \qquad ({\sim}_{\mathrm{def}})$$
$$\mathbf{0} \mapsto \mathbf{0}$$
$$\mathbf{1} \mapsto \mathbf{1}$$

is an interpretation of $\mathcal{N}$ *in* $\mathcal{NFL}_{ew}$.

3. *The interpretations* δ *and* ε *are mutually inverse.*

Hence the varieties of Nelson algebras and Nelson FL_{ew}*-algebras are term equivalent.*

2.3. k-deductive systems

Let Λ be a language type and let $1 \leq k < \omega$. A *k-formula* is an element of the Cartesian product Fm_Λ^k. We denote k-formulas using lowercase boldface Greek letters $\boldsymbol{\varphi}, \boldsymbol{\psi}, \ldots$, except when $k = 1$, where we write simply $\varphi, \psi, \ldots$. Given a substitution $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ and a k-formula $\boldsymbol{\varphi} := \langle \varphi_0, \ldots, \varphi_{k-1} \rangle$, we write variously $\sigma\boldsymbol{\varphi}$ or $\sigma(\boldsymbol{\varphi})$ for $\big\langle \sigma(\varphi_0), \ldots, \sigma(\varphi_{k-1}) \big\rangle$. For $\Gamma \subseteq \mathrm{Fm}_\Lambda^k$ we write $\sigma(\Gamma)$ for $\big\{\sigma(\boldsymbol{\varphi}) : \boldsymbol{\varphi} \in \Gamma\big\}$.

A *k-deductive system* is a pair $\mathbf{S} := \langle \Lambda, \vdash_{\mathbf{S}} \rangle$, where $\vdash_{\mathbf{S}} \subseteq \wp(\mathrm{Fm}_\Lambda^k) \times \mathrm{Fm}_\Lambda^k$, and the following conditions are satisfied for all $\Gamma, \Delta \subseteq \mathrm{Fm}_\Lambda^k$ and $\boldsymbol{\varphi} \in \mathrm{Fm}_\Lambda^k$ [6, Definition 3.1]:

1. $\varphi \in \Gamma$ implies $\Gamma \vdash_{\mathbf{S}} \varphi$;
2. $\Gamma \vdash_{\mathbf{S}} \varphi$ and $\Delta \vdash_{\mathbf{S}} \psi$ for every $\psi \in \Gamma$ implies $\Delta \vdash_{\mathbf{S}} \varphi$;
3. $\Gamma \vdash_{\mathbf{S}} \varphi$ implies $\Gamma' \vdash_{\mathbf{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$;
4. $\Gamma \vdash_{\mathbf{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathbf{S}} \sigma(\varphi)$ for every substitution σ.

A *deductive system* is a 1-deductive system.

Let $\mathbf{S}$ be a k-deductive system. The relation $\vdash_{\mathbf{S}}$ is called the *consequence relation* of $\mathbf{S}$. The *consequence operator* associated with $\vdash_{\mathbf{S}}$ is the map $\mathrm{Cn}_{\mathbf{S}} : \wp(\mathrm{Fm}^k_\Lambda) \to \wp(\mathrm{Fm}^k_\Lambda)$ given by $\mathrm{Cn}_{\mathbf{S}}(\Gamma) := \{\varphi \in \mathrm{Fm}^k_\Lambda : \Gamma \vdash_{\mathbf{S}} \varphi\}$. A set $T \subseteq \mathrm{Fm}^k_\Lambda$ is called an $\mathbf{S}$-*theory* (briefly, a *theory*) if $T \vdash_{\mathbf{S}} \varphi$ implies $\varphi \in T$, for each $\varphi \in \mathrm{Fm}^k_\Lambda$. The set of all theories of $\mathbf{S}$ is denoted $\mathrm{Th}\,\mathbf{S}$. For $\Gamma, \Delta \subseteq \mathrm{Fm}^k_\Lambda$, the notation $\Gamma \vdash_{\mathbf{S}} \Delta$ abbreviates '$\Gamma \vdash_{\mathbf{S}} \varphi$ for all $\varphi \in \Delta$', while $\Gamma \dashv\vdash_{\mathbf{S}} \Delta$ abbreviates 'both $\Gamma \vdash_{\mathbf{S}} \Delta$ and $\Delta \vdash_{\mathbf{S}} \Gamma$'. For a systematic exposition of the theory of k-deductive systems, see Blok and Pigozzi [4, 6].

2.4. Definitional equivalence for k-deductive systems

Let $\mathbf{A} := \langle A; c^{\mathbf{A}} \rangle_{c \in \Lambda}$ be an algebra of type Λ, and let $F \subseteq A^k$ for $k \geq 1$. A congruence θ on $\mathbf{A}$ is said to be *compatible* with F if $\langle a_0, \ldots, a_{k-1} \rangle \in F$ and $a_i \;\theta\; b_i$ $(i = 0, \ldots, k-1)$ imply $\langle b_0, \ldots, b_{k-1} \rangle \in F$. The *Leibniz congruence on* $\mathbf{A}$ *over* F is the largest congruence on $\mathbf{A}$ compatible with F. In symbols,

$$\Omega^{\mathbf{A}} F := \bigvee \{\theta \in \mathrm{Con}\,\mathbf{A} : \theta \text{ is compatible with } F\}.$$

We write simply Ω for $\Omega^{\mathbf{Fm}_\Lambda}$. For a survey of the operator $\Omega^{\mathbf{A}} F$ in abstract algebraic logic, see [16].

For a k-dimensional deductive system $\mathbf{S}$, the *Tarski congruence* $\widetilde{\Omega}(\mathbf{S})$ is the largest congruence on the formula algebra that is compatible with every theory of $\mathbf{S}$. In symbols,

$$\widetilde{\Omega}(\mathbf{S}) := \bigcap \{\Omega\, T : T \in \mathrm{Th}\,\mathbf{S}\}.$$

For studies of the Tarski congruence in (second-order) abstract algebraic logic see [17, 15].

Let Λ_1 and Λ_2 be two language types, and let α be a map from Λ_1 to Fm_{Λ_2}. The *standard extension* of α is the function $\bar{\alpha} : \mathrm{Fm}_{\Lambda_1} \to \mathrm{Fm}_{\Lambda_2}$ defined recursively based on the complexity of terms by:

$$(v_i)^{\bar{\alpha}} = v_i,$$
$$(c\varphi_0, \ldots, \varphi_{n-1})^{\bar{\alpha}} = [\![\varphi_0^{\bar{\alpha}}, \ldots, \varphi_{n-1}^{\bar{\alpha}}]\!] c^{\alpha}$$

where v_i is a variable, $c \in \Lambda_1$ is an n-ary connective, $\varphi_0, \ldots, \varphi_{n-1}$ are Λ_1-formulas, and $[\![\varphi_0, \ldots, \varphi_{n-1}]\!]$ is the surjective substitution that takes values φ_i on v_i for $i = 0, \ldots, n-1$, and takes value v_i on v_{i+n} [21, Section 2.1.1, p. 48]. The map $\bar{\alpha}$ extends to k-formulas in the natural way on defining $\boldsymbol{\varphi}^{\bar{\alpha}} := \langle \varphi_0^{\bar{\alpha}}, \ldots, \varphi_{k-1}^{\bar{\alpha}} \rangle$ for all $\boldsymbol{\varphi} := \langle \varphi_0, \ldots, \varphi_{k-1} \rangle \in \mathrm{Fm}^k_{\Lambda_1}$ and $\Gamma^{\bar{\alpha}} := \{\boldsymbol{\varphi}^{\bar{\alpha}} : \boldsymbol{\varphi} \in \Gamma\}$ for all $\Gamma \subseteq \mathrm{Fm}^k_{\Lambda_1}$.

Let $\mathbf{S}_1 := \langle \Lambda_1, \vdash_{\mathbf{S}_1} \rangle$ and $\mathbf{S}_2 := \langle \Lambda_2, \vdash_{\mathbf{S}_2} \rangle$ be two k-dimensional deductive systems. A mapping $\alpha : \Lambda_1 \to \mathrm{Fm}_{\Lambda_2}$ is said to be an *interpretation* of $\mathbf{S}_1$ in $\mathbf{S}_2$ if it satisfies the following two conditions [21, Definition 2.5]:

(DE-1) $\langle c^\alpha, \mu c^\alpha \rangle \in \widetilde{\Omega}(\mathbf{S}_2)$ for all connectives c of Λ_1 with arity n and substitutions μ of Λ_2 that fix the first n variables;

(DE-2) If $\Gamma \vdash_{\mathbf{S}_1} \boldsymbol{\varphi}$ then $\Gamma^{\bar{\alpha}} \vdash_{\mathbf{S}_2} \boldsymbol{\varphi}^{\bar{\alpha}}$ for all $\Gamma \subseteq \mathrm{Fm}^k_{\Lambda_1}$ and $\boldsymbol{\varphi} \in \mathrm{Fm}^k_{\Lambda_1}$.

Let α be an interpretation of $\mathbf{S}_1$ in $\mathbf{S}_2$, and β an interpretation of $\mathbf{S}_2$ in $\mathbf{S}_1$. We say that α and β are *mutually inverse* if $\langle \varphi, \varphi^{\bar{\alpha}\bar{\beta}} \rangle \in \widetilde{\Omega}(\mathbf{S}_1)$ and $\langle \psi, \psi^{\bar{\beta}\bar{\alpha}} \rangle \in \widetilde{\Omega}(\mathbf{S}_2)$ for all $\varphi \in \mathrm{Fm}_{\Lambda_1}$ and $\psi \in \mathrm{Fm}_{\Lambda_2}$. The deductive systems $\mathbf{S}_1$ and $\mathbf{S}_2$ are said to be *definitionally equivalent* if there are interpretations α of $\mathbf{S}_1$ in $\mathbf{S}_2$ and β of $\mathbf{S}_2$ in $\mathbf{S}_1$ that are mutually inverse [21, Definition 2.14].[5]

The notion of definitional equivalence for k-deductive systems presented here is due to Gyuris [21]. For alternative notions of definitional equivalence with applicability to abstract algebraic logic see [47, 32, 12]. For a comparison between the notion of definitional equivalence presented here and the notion of equipollence [12] due to Caleiro and Gonçalves, see [39].

2.5. Regularly algebraisable logics

Let $\mathbf{S}$ be a deductive system over a language type Λ. Recall from [15, Section 1.4, p. 36] that a finite set $\{\Delta_0, \ldots, \Delta_{m-1}\}$ of Λ-formulas in two variables is a *finite system of equivalence formulas* for $\mathbf{S}$ if for any n-ary connective $c \in \Lambda$ and any set of Λ-formulas $\{\varphi_k : k = 0, \ldots, n-1\} \cup \{\psi_k : k = 0, \ldots, n-1\} \cup \{\varphi, \psi, \chi\}$ the following conditions hold for $j = 0, \ldots, m-1$:

(ALG1) $\vdash_{\mathbf{S}} \varphi \, \Delta_j \, \varphi$[6]

(ALG2) $\varphi, \{\varphi \, \Delta_i \, \psi : i = 0, \ldots, m-1\} \vdash_{\mathbf{S}} \psi$

(ALG3) $\{\varphi \, \Delta_i \, \psi : i = 0, \ldots, m-1\} \vdash_{\mathbf{S}} \psi \, \Delta_j \, \varphi$

---

[5] For a discussion of the distinction between definitional equivalence as described in this paper, and the more familiar notion in algebraic logic of deductive equivalence, see Blok and Pigozzi [6, Note 4.1].

[6] To simplify notation, we are writing $\varphi \, \Delta_j \, \varphi$ for $\Delta_j(\varphi, \varphi)$, etc., here and in the sequel.

(ALG4) $\{\varphi\,\Delta_i\,\psi : i = 0,\dots,m-1\}, \{\psi\,\Delta_i\,\chi : i = 0,\dots,m-1\} \vdash_{\mathbf{S}} \varphi\,\Delta_j\,\chi$

(ALG5) $\{\varphi_k\,\Delta_i\,\psi_k : i = 0,\dots,m-1;\ k = 0,\dots,n-1\} \vdash_{\mathbf{S}}$
$c(\varphi_0,\dots,\varphi_{n-1})\,\Delta_j\,c(\psi_0,\dots,\psi_{n-1})$.

S is said to be *regularly algebraisable* if it has a finite system of equivalence formulas and in addition the following conditions hold for $j = 0,\dots,m-1$:

(ALG6) $\varphi, \psi \vdash_{\mathbf{S}} \varphi\,\Delta_j\,\psi$.

By [15, Theorem 28], every regularly algebraisable logic is algebraisable in the sense of Blok and Pigozzi [3]. For studies of regularly algebraisable logics, see [34, 14, 15].

Let **S** be a regularly algebraisable deductive system over a language type Λ with finite system of equivalence formulas $\{\Delta_j : j = 0,\dots,m-1\}$. Then there exists a unique quasivariety $\mathsf{Alg\,Mod}^*\,\mathbf{S}$ of algebras of type Λ, and a constant term $\mathbf{1} := \Delta_j(x,x)$ of $\mathsf{Alg\,Mod}^*\,\mathbf{S}$, such that the following conditions hold for any $\Gamma \cup \{\varphi,\psi\} \subseteq \mathrm{Fm}_\Lambda$:[7]

(EQV1) $\Gamma \vdash_{\mathbf{S}} \varphi \quad$ if and only if $\quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{Alg\,Mod}^*\,\mathbf{S}} \varphi \approx \mathbf{1}$

(EQV2) $\varphi \approx \psi \mathrel{=\!\!|\!\!\models}_{\mathsf{Alg\,Mod}^*\,\mathbf{S}} \{\varphi\,\Delta_j\,\psi \approx \mathbf{1} : j = 0,\dots,m-1\}$.

(Here $\Gamma \mathrel{=\!\!|\!\!\models}_{\mathsf{Alg\,Mod}^*\,\mathbf{S}} \Gamma'$ abbreviates '$\Gamma \models_{\mathsf{Alg\,Mod}^*\,\mathbf{S}} \Gamma'$ and $\Gamma' \models_{\mathsf{Alg\,Mod}^*\,\mathbf{S}} \Gamma$'.) The class $\mathsf{Alg\,Mod}^*\,\mathbf{S}$ is called the *equivalent quasivariety semantics* of **S**. For any presentation of **S** by a set of axioms Ax and (proper) inference rules Ru, the equivalent quasivariety $\mathsf{Alg\,Mod}^*\,\mathbf{S}$ is determined by the following collection of identities and quasi-identities [15, Theorem 30]:

(AX-1) $\varphi \approx \mathbf{1}$, for each $\varphi \in Ax$

(AX-2) $\psi_0 \approx \mathbf{1}$ and $\dots$ and $\psi_{p-1} \approx \mathbf{1}$ implies $\varphi \approx \mathbf{1}$
for each inference rule $\langle \psi_0,\dots,\psi_{p-1},\varphi\rangle \in Ru$

(AX-3) $\Delta_0(x,y) \approx \mathbf{1}$ and $\dots$ and $\Delta_{m-1}(x,y) \approx \mathbf{1}$ implies $x \approx y$.

The remarks of this section extend in a natural way to deductive systems that are *algebraisable* in the sense of Blok and Pigozzi [3]. For details, see [3, 6, 14]. For all other terminology and notation of abstract algebraic logic not specified either above or in the sequel see Czelakowski and Pigozzi [15] and Blok and Pigozzi [3, 6].

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[7] By [44, Theorem 3.2.4, p. 182], $\mathsf{Alg\,Mod}^*\,\mathbf{S} \models \Delta_i(x,x) \approx \Delta_{i'}(y,y)$ for all $0 \le i, i' \le m-1$. Hence $\Delta_j(x,x)$ is a constant term of $\mathsf{Alg\,Mod}^*\,\mathbf{S}$ as claimed.

2.6. 1-assertional logics

Let $\mathcal{K}$ be a pointed quasivariety over a language type Λ. The $\mathbf{1}$-*assertional logic* of $\mathcal{K}$, in symbols $\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}$, is the deductive system from sets of Λ-terms to Λ-terms determined by the equivalence [15, Corollary 33]:

$$\Gamma \vdash_{\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}} \varphi \quad \text{if and only if} \quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathcal{K}} \varphi \approx \mathbf{1}$$

for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\Lambda$.[8] For studies of assertional logics see [14, 15, 7].

A pointed quasivariety $\mathcal{K}$ is said to be *relatively point regular* if, whenever $\mathbf{A} \in \mathcal{K}$ and $\theta, \phi \in \mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$ with $\mathbf{1}^{\mathbf{A}}/\theta = \mathbf{1}^{\mathbf{A}}/\phi$, we have that $\theta = \phi$. The following result of Czelakowski and Pigozzi [15] exhibits a one-one correspondence between regularly algebraisable logics and relatively point regular quasivarieties.

Theorem 2.2. [15, Corollary 35]

1. *Every regularly algebraisable deductive system* $\mathbf{S}$ *is the* $\mathbf{1}$*-assertional logic of a unique relatively point regular quasivariety, namely its equivalent quasivariety semantics. In symbols,* $\mathbf{S} = \mathbf{S}^{\mathrm{ASL}}\,\mathsf{Alg}\,\mathsf{Mod}^*\,\mathbf{S}$.
2. *Every relatively point regular quasivariety* $\mathcal{K}$ *is the equivalent quasivariety semantics of a unique regularly algebraisable deductive system, namely its* $\mathbf{1}$*-assertional logic. In symbols,* $\mathcal{K} = \mathsf{Alg}\,\mathsf{Mod}^*\,\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}$.

3. Substructural logics over FL

In this section we briefly criticise the notion of substructural logic over $\mathbf{FL}_S$ ($S \subseteq \{e, c, i, o\}$) presented in [18, 19] from the perspective of algebraic and non-classical logic, propose an alternative definition, and characterise (in the sense of this paper) the substructural logics over $\mathbf{FL}_S$.

According to Galatos and Ono [18, Section 3.1, p. 285], and Galatos *et al.* [19, Section 2.1.4], a substructural logic over $\mathbf{FL}_S$ is a theory of $\mathbf{FL}_S$ closed under substitutions, or equivalently, the set of theorems of an axiomatic extension of $\mathbf{FL}_S$. This definition is unorthodox in that:

- Deductive systems are viewed as sets of formulas and *not* as consequence relations. The study of substructural logics over **FL** in the sense of [18, 19] thereby amounts to an investigation, in the framework of the

[8] Since $\mathcal{K}$ is closed under the formation of ultraproducts, $\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}$ is finitary and hence is a deductive system in the sense of this paper.

Blok-Pigozzi theory of algebraisable logics [3], of the *axiomatic* extensions of **FL** via an examination of the *subvarieties* of the variety of FL-algebras. But in full generality, the study of an algebraisable deductive system **S** is tantamount to an investigation of the *extensions* of **S** via an examination of the *subquasivarieties* of its equivalent quasivariety semantics. (A justification for these remarks is given prior to the statement of Corollary 3.2 below.) Thus the definition of substructural logic over $\mathbf{FL}_S$ due to [18, 19] is in a sense unduly restrictive.

- There is nothing that prohibits a logic having *all* the structural rules from being substructural. Indeed, the classical propositional calculus is a substructural logic over $\mathbf{FL}_{ecw}$ in the sense of [18, 19], as Galatos and Ono explicitly point out in [18, p. 279]. But, as Restall [35, p. 1] asserts, "Substructural logics [should] focus on the behaviour and presence — or more suggestively, the *absence* — of *structural rules*" [italics Restall's].[9] Thus the definition of substructural logic over $\mathbf{FL}_S$ due to [18, 19] is in a sense overly generous.

Let **S** be a deductive system over a language type Λ. An *extension* of **S** is any system $\mathbf{S}' := \langle \Lambda, \vdash_{\mathbf{S}'} \rangle$ over the same language type Λ such that $\Gamma \vdash_{\mathbf{S}} \varphi$ implies $\Gamma \vdash_{\mathbf{S}'} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\Lambda$. $\mathbf{S}'$ is said to be *axiomatic* if it can be obtained by adjoining new axioms to **S** only. By Blok and Pigozzi [3, Corollary 4.9], any extension of a (regularly) algebraisable deductive system is itself (regularly) algebraisable.

A deductive system **S** over a language type Λ is said to be *Fregean* if, for every $T \in \mathrm{Th}\,\mathbf{S}$, the relativised interderivability relation $\dashv\vdash^T_{\mathbf{S}}$ defined for all $\varphi, \psi \in \mathrm{Fm}_\Lambda$ by

$$\varphi \dashv\vdash^T_{\mathbf{S}} \psi \quad \text{if and only if} \quad T, \varphi \vdash_{\mathbf{S}} \psi \text{ and } T, \psi \vdash_{\mathbf{S}} \varphi$$

is a congruence relation on $\mathbf{Fm}_\Lambda$ [15, Definition 59]. **S** is *non-Fregean* if it is not Fregean. For studies of Fregean logics, see [17, 14, 15].

The discussion heading this section leads us to the following definition. A *substructural logic over* $\mathbf{FL}_S$, $S \subseteq \{e, c, i, o\}$, is a deductive system **S** that is definitionally equivalent to a non-Fregean extension of $\mathbf{FL}_S$. The next result shows the notion of substructural logic over $\mathbf{FL}_S$ used in this paper appropriately captures the notion of a substructural logic over **FL** as an extension of **FL** lacking some or all of the structural rules.

[9] For further support for this point of view see e.g. Došen [36, p. 6].

Theorem 3.1. *An extension* $\mathbf{S}$ *of* $\mathbf{FL}$ *is Fregean if and only if it is an axiomatic extension of* $\mathbf{FL}_{eci}$.[10]

Proof. It is clear that any axiomatic extension of $\mathbf{FL}_{eci}$ is Fregean. For the converse, suppose $\mathbf{S}$ is a Fregean extension of $\mathbf{FL}$. Because $\mathbf{S}$ is Fregean and algebraisable with theorems, from Czelakowski and Pigozzi [15, Theorem 61] we have that $\mathbf{S}$ is regularly algebraisable. Since $\mathbf{S}$ is regularly algebraisable, $p \approx \mathbf{1}$ is a single defining equation for $\mathbf{S}$ in the sense of Blok and Pigozzi [3, Definition 2.2]. This implies that $\mathbf{S}$ is an extension of the deductive system $\mathbf{FL}_i$.

Observe next that $\{p \backslash q\}$ is a protoequivalence system for $\mathbf{S}$ in the sense of Czelakowski and Pigozzi [15, Section 1.4, p. 32]. Since $\wedge$ is a conjunction formula for $\mathbf{S}$ in the sense of [15, Section 2.2, p. 57], and $\mathbf{S}$ is Fregean and algebraisable with theorems, from [15, Theorem 64] we have that $\mathbf{S}$ has the uniterm deduction-detachment theorem (in the sense of [15, Definition 38]) with uniterm deduction-detachment system $\{p \backslash (p \wedge q)\}$. Because $\mathbf{S}$ is an extension of $\mathbf{FL}_i$, the formulas $\varphi \backslash (\varphi \wedge \psi)$ and $\varphi \backslash \psi$ are provably equivalent (in the sense of [19, Section 2.1.2]) over $\mathbf{S}$. Therefore $\{p \backslash q\}$ is also a uniterm deduction-detachment system for $\mathbf{S}$. This suffices to guarantee that $\mathbf{S}$ is an extension of $\mathbf{FL}_{eci}$.

It remains only to observe that $\mathbf{S}$ is an axiomatic extension of $\mathbf{FL}_{eci}$. Because $\wedge$ is a conjunction formula for $\mathbf{S}$, the deductive system $\mathbf{S}$ has the property of conjunction in the sense of Font and Jansana [17, Definition 2.45]. Since $\mathbf{S}$ is Fregean and algebraisable with theorems, from Font and Jansana [17, Corollary 4.32] we have that $\mathbf{S}$ is strongly algebraisable (*i.e.*, $\mathsf{Alg}\,\mathsf{Mod}^* \mathbf{S}$ is a variety). The claim that $\mathbf{S}$ is an axiomatic extension of $\mathbf{FL}_{eci}$ now follows, because $\mathbf{S}$ is regularly algebraisable. ∎

A pointed quasivariety $\mathcal{K}$ is said to be *relatively congruence orderable* if, for every $\mathbf{A} \in \mathcal{K}$ and all $a, b \in A$, $\Theta^{\mathbf{A}}_{\mathcal{K}}(a, \mathbf{1}^{\mathbf{A}}) = \Theta^{\mathbf{A}}_{\mathcal{K}}(b, \mathbf{1}^{\mathbf{A}})$ implies $a = b$. $\mathcal{K}$ is said to be *Fregean* if it is both relatively point regular and relatively congruence orderable [15, Definition 85]. For studies of Fregean quasivarieties in general algebra, see [31, 1, 22].

By [27, Corollary 1.3.5], there exists a lattice anti-isomorphism from the lattice of extensions of an algebraisable deductive system $\mathbf{S}$ onto the lattice of subquasivarieties of $\mathsf{Alg}\,\mathsf{Mod}^* \mathbf{S}$, which moreover maps each extension of $\mathbf{S}$ to its equivalent quasivariety. Combining these remarks with Theorem 2.2, Theorem 3.1, and Czelakowski and Pigozzi [15, Theorem 86] yields

[10] The deductive system $\mathbf{FL}_{eci}$ is definitionally equivalent to Johansson's minimal logic [34, Chapter XI], [47, Section 2.7]. For a discussion, see [19, Section 2.3.8].

the following corollary, which is due independently to the first author and to N. Galatos (unpublished).

COROLLARY 3.2. *A quasivariety of FL-algebras is Fregean if and only if it is a variety of* FL_{eci}*-algebras.*[11]

For recent results related to Theorem 3.1 and Corollary 3.2, see Bou *et al.* [9, Section 4].

4. Definitional equivalence for regularly algebraisable logics

In this section we give a sufficient condition for two regularly algebraisable logics to be definitionally equivalent (Theorem 4.6).

Let $\mathcal{K}$ be a quasivariety over a language type Λ axiomatised by a set of identities Id and a set of quasi-identities QId. Recall from Czelakowski and Pigozzi [15, Definition 2] or Blok and Pigozzi [6, Section 3.3.2] that the *applied equational logic* determined by $\mathcal{K}$, in symbols $\mathbf{S}^{\mathrm{EQL}}\mathcal{K}$, is the 2-dimensional deductive system presented by the following collection of axioms and inference rules:

(EQ-1) $\langle p, p\rangle$

(EQ-2) $\dfrac{\langle p, q\rangle}{\langle q, p\rangle}$

(EQ-3) $\dfrac{\langle p, q\rangle, \langle q, r\rangle}{\langle p, r\rangle}$

(EQ-4) $\dfrac{\langle p_0, q_0\rangle, \ldots, \langle p_{n-1}, q_{n-1}\rangle}{\big\langle c(p_0, \ldots, p_{n-1}), c(q_0, \ldots, q_{n-1})\big\rangle}$ for each $c \in \Lambda$ of arity n

(EQ-5) $\langle \varphi, \psi\rangle$ for every identity $\forall \bar{x}(\varphi \approx \psi) \in Id$

(EQ-6) $\dfrac{\langle \chi_0, \zeta_0\rangle, \ldots, \langle \chi_{n-1}, \zeta_{n-1}\rangle}{\langle \varphi, \psi\rangle}$ for every quasi-identity

$$\forall \bar{x}(\chi_0 \approx \zeta_0 \text{ and } \ldots \text{ and } \chi_{n-1} \approx \zeta_{n-1} \text{ implies } \varphi \approx \psi) \in QId.$$

Applied equational logics have the following

THEOREM 4.1 (Completeness theorem). [6, Theorem 3.9] *Let* $\mathcal{K}$ *be a quasivariety over a language type* Λ *and let* $\Gamma \cup \big\{\langle \varphi_0, \varphi_1\rangle\big\} \subseteq \mathrm{Fm}^2_\Lambda$*. Then*

$$\big\{\langle \psi_0, \psi_1\rangle : \langle \psi_0, \psi_1\rangle \in \Gamma\big\} \vdash_{\mathbf{S}^{\mathrm{EQL}}\mathcal{K}} \langle \varphi_0, \varphi_1\rangle \quad \textit{if and only if}$$
$$\big\{\psi_0 \approx \psi_1 : \langle \psi_0, \psi_1\rangle \in \Gamma\big\} \models_{\mathcal{K}} \varphi_0 \approx \varphi_1.$$

[11] The variety of FL_{eci}-algebras is term equivalent to the variety of generalised Heyting algebras. For a discussion, see [19, Section 2.3.8] or [9, Section 4].

Let $\mathbf{S}$ be a deductive system. The following useful technical lemma of Czelakowski and Pigozzi [15] asserts that the $\mathsf{Alg\,Mod}^*\,\mathbf{S}$-congruences on the formula algebra are precisely the Leibniz congruences.

LEMMA 4.2. [15, Lemma 12] *Let $\mathbf{S}$ be a deductive system over a language type Λ. Then $\mathrm{Con}_{\mathsf{Alg\,Mod}^*\,\mathbf{S}}\,\mathbf{Fm}_\Lambda = \{\Omega\, T : T \in \mathrm{Th}\,\mathbf{S}\}$.*

For an applied equational logic $\mathbf{S}$, $\widetilde{\Omega}(\mathbf{S})$ has a particularly transparent description:

LEMMA 4.3. [21, Proposition 1.26] *Let $\mathbf{S}$ be an applied equational logic. Then $\widetilde{\Omega}(\mathbf{S}) = \mathrm{Cn}_{\mathbf{S}}(\emptyset)$.*

The next result, due to Gyuris, shows that the notion of definitional equivalence for deductive systems generalises the notion of term equivalence for quasivarieties described in Part I of this series [41, Section 2.1].

PROPOSITION 4.4. [21, Proposition 2.17] *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two quasivarieties over language types Λ_1 and Λ_2. Let $\mathbf{S}_1 := \mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_1$ and $\mathbf{S}_2 := \mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_2$ be the applied equational logics determined by $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then $\mathbf{S}_1$ and $\mathbf{S}_2$ are definitionally equivalent if and only if $\mathcal{K}_1$ and $\mathcal{K}_2$ are term equivalent. In particular, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are term equivalent with interpretations $\alpha : \Lambda_1 \to \mathrm{Fm}_{\Lambda_2}$ and $\beta : \Lambda_2 \to \mathrm{Fm}_{\Lambda_1}$ then $\mathbf{S}_1$ and $\mathbf{S}_2$ are definitionally equivalent with the same mutually inverse interpretations.*

In Theorem 4.6 below, we lift the right-to-left direction of Proposition 4.4 to the setting of regularly algebraisable logics. But first, a technical lemma.

LEMMA 4.5. *Let $\mathcal{K}$ be a relatively point regular quasivariety over a language type Λ. If $\varphi \equiv \psi \pmod{\widetilde{\Omega}(\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K})}$ for Λ-formulas φ, ψ, then $\varphi \equiv \psi \pmod{\widetilde{\Omega}(\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K})}$.*

PROOF. Suppose $\varphi \equiv \psi \pmod{\widetilde{\Omega}(\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K})}$. By Lemma 4.3, $\vdash_{\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}} \langle \varphi, \psi \rangle$. By the completeness theorem for applied equational logics, therefore, we have that $\mathcal{K} \models \varphi \approx \psi$, whence $\mathbf{Fm}_\Lambda/\theta \models \varphi \approx \psi$ for all $\theta \in \mathrm{Con}_{\mathcal{K}}\,\mathbf{Fm}_\Lambda$. Since $\mathrm{Con}_{\mathsf{Alg\,Mod}^*\,\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}}\,\mathbf{Fm}_\Lambda = \mathrm{Con}_{\mathcal{K}}\,\mathbf{Fm}_\Lambda$ (by Theorem 2.2), we conclude that $\varphi \equiv \psi \pmod{\theta}$ for all $\theta \in \mathrm{Con}_{\mathsf{Alg\,Mod}^*\,\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}}\,\mathbf{Fm}_\Lambda$. By Lemma 4.2, therefore, $\varphi \equiv \psi \pmod{\Omega\, T}$ for all $T \in \mathrm{Th}\,\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}$. Thus $\varphi \equiv \psi \pmod{\bigcap\{\Omega\, T : T \in \mathrm{Th}\,\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}\}}$, which is to say $\varphi \equiv \psi \pmod{\widetilde{\Omega}(\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K})}$ as claimed. ■

THEOREM 4.6. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two regularly algebraisable deductive systems over language types Λ_1 and Λ_2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the relatively $\mathbf{1}^{\mathcal{K}_1}$-regular and relatively $\mathbf{1}^{\mathcal{K}_2}$-regular quasivarieties comprising the equivalent quasivariety semantics of $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are term* *equivalent with interpretations $\alpha : \Lambda_1 \to \mathrm{Fm}_{\Lambda_2}$ and $\beta : \Lambda_2 \to \mathrm{Fm}_{\Lambda_1}$ such that $(\mathbf{1}^{\mathcal{K}_1})^\alpha = \mathbf{1}^{\mathcal{K}_2}$ and $(\mathbf{1}^{\mathcal{K}_2})^\beta = \mathbf{1}^{\mathcal{K}_1}$. Then $\mathbf{S}_1$ and $\mathbf{S}_2$ are definitionally equivalent with the same mutually inverse interpretations.*

PROOF. By Proposition 4.4, $\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_1$ and $\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_2$ are definitionally equivalent with mutually inverse interpretations $\alpha : \Lambda_1 \to \mathrm{Fm}_{\Lambda_2}$ and $\beta : \Lambda_2 \to \mathrm{Fm}_{\Lambda_1}$. Throughout the proof we make implicit use of this observation.

Let c be an n-ary basic connective of Λ_1 and μ a substitution of Λ_2 that fixes the first n variables. By (DE-1), $\langle c^\alpha, \mu c^\alpha \rangle \in \widetilde{\Omega}(\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_2)$, so by Lemma 4.5, $\langle c^\alpha, \mu c^\alpha \rangle \in \widetilde{\Omega}(\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}_2)$. By Theorem 2.2(1), we conclude that $\langle c^\alpha, \mu c^\alpha \rangle \in \widetilde{\Omega}(\mathbf{S}_2)$. Observe next that for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\Lambda_1}$,

$$
\begin{array}{rll}
\Gamma \vdash_{\mathbf{S}_1} \varphi \quad \text{iff} & \Gamma \vdash_{\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}_1} \varphi & \text{by Theorem 2.2} \\
\text{iff} & \{\psi \approx \mathbf{1}^{\mathcal{K}_1} : \psi \in \Gamma\} \models_{\mathcal{K}_1} \varphi \approx \mathbf{1}^{\mathcal{K}_1} & \\
\text{iff} & \big\{\langle \psi, \mathbf{1}^{\mathcal{K}_1}\rangle : \psi \in \Gamma\big\} \vdash_{\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_1} \langle \varphi, \mathbf{1}^{\mathcal{K}_1}\rangle & \text{by Theorem 4.1} \\
\text{only if} & \big\{\langle \psi^{\bar\alpha}, (\mathbf{1}^{\mathcal{K}_1})^{\bar\alpha}\rangle : \psi \in \Gamma\big\} \vdash_{\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_2} \langle \varphi^{\bar\alpha}, (\mathbf{1}^{\mathcal{K}_1})^{\bar\alpha}\rangle & \text{by (DE-2)} \\
\text{iff} & \big\{\langle \psi^{\bar\alpha}, \mathbf{1}^{\mathcal{K}_2} : \psi \in \Gamma\rangle\big\} \vdash_{\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_2} \langle \varphi^{\bar\alpha}, \mathbf{1}^{\mathcal{K}_2}\rangle & \\
\text{iff} & \{\psi^{\bar\alpha} \approx \mathbf{1}^{\mathcal{K}_2} : \psi \in \Gamma\} \models_{\mathcal{K}_2} \varphi^{\bar\alpha} \approx \mathbf{1}^{\mathcal{K}_2} & \text{by Theorem 4.1} \\
\text{iff} & \Gamma^{\bar\alpha} \vdash_{\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}_2} \varphi^{\bar\alpha} & \\
\text{iff} & \Gamma^{\bar\alpha} \vdash_{\mathbf{S}_2} \varphi^{\bar\alpha} & \text{by Theorem 2.2.}
\end{array}
$$

This shows that α is an interpretation of $\mathbf{S}_1$ in $\mathbf{S}_2$. A similar argument verifies that β is an interpretation of $\mathbf{S}_2$ in $\mathbf{S}_1$.

Since $\langle \varphi, \varphi^{\bar\alpha\bar\beta}\rangle \in \widetilde{\Omega}(\mathbf{S}^{\mathrm{EQL}}\,\mathcal{K}_1)$ for any $\varphi \in \mathrm{Fm}_{\Lambda_1}$, we have that $\langle \varphi, \varphi^{\bar\alpha\bar\beta}\rangle \in \widetilde{\Omega}(\mathbf{S}^{\mathrm{ASL}}\,\mathcal{K}_1)$ by Lemma 4.5. By Theorem 2.2(1), $\langle \varphi, \varphi^{\bar\alpha\bar\beta}\rangle \in \widetilde{\Omega}(\mathbf{S}_1)$. A similar argument establishes $\langle \varphi, \varphi^{\bar\beta\bar\alpha}\rangle \in \widetilde{\Omega}(\mathbf{S}_2)$ for any $\varphi \in \mathrm{Fm}_{\Lambda_2}$. Hence the interpretations α and β are mutually inverse. This completes the proof that $\mathbf{S}_1$ and $\mathbf{S}_2$ are definitionally equivalent. ■

5. N is a substructural logic over $\mathbf{FL}_{ew}$

In this section we complete the proofs of Theorems 1.1 and 1.2. We give a (Hilbert-style) axiomatisation of a certain deductive system $\mathbf{H}$, and show that $\mathbf{H}$ is $\mathbf{FL}_{ew}$ (Lemma 5.4). We present $\mathbf{NFL}_{ew}$ as an axiomatic extension of $\mathbf{H}$, and verify that its equivalent variety semantics is $\mathcal{NFL}_{ew}$ (Corollary 5.6). From the term equivalence of the varieties $\mathcal{NFL}_{ew}$ and $\mathcal{N}$ (Theorem 2.1), we conclude that the deductive systems $\mathbf{NFL}_{ew}$ and $\mathbf{N}$ are definitionally equivalent (Theorem 1.1). It follows from this observation that $\mathbf{N}$ is a substructural logic over $\mathbf{FL}_{ew}$ (Theorem 1.2).

Let $\mathbf{H}$ denote the deductive system over the language type $\Lambda[\mathbf{FL}_{ew}]$ presented by the following collection of axioms and inference rules:[12]

$$(p \Rightarrow q) \Rightarrow \big((q \Rightarrow r) \Rightarrow (p \Rightarrow r)\big) \tag{A1}$$
$$\big(p \Rightarrow (q \Rightarrow r)\big) \Rightarrow \big(q \Rightarrow (p \Rightarrow r)\big) \tag{A2}$$
$$p \Rightarrow (q \Rightarrow p) \tag{A3}$$
$$p \Rightarrow \big(q \Rightarrow (p * q)\big) \tag{A4}$$
$$\big(p \Rightarrow (q \Rightarrow r)\big) \Rightarrow \big((p * q) \Rightarrow r\big) \tag{A5}$$
$$(p \wedge q) \Rightarrow p \tag{A6}$$
$$(p \wedge q) \Rightarrow q \tag{A7}$$
$$(p \Rightarrow q) \Rightarrow \big((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r))\big) \tag{A8}$$
$$p \Rightarrow (p \vee q) \tag{A9}$$
$$q \Rightarrow (p \vee q) \tag{A10}$$
$$(p \Rightarrow r) \Rightarrow \big((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)\big) \tag{A11}$$
$$\mathbf{1} \tag{A12}$$
$$\mathbf{0} \Rightarrow p \tag{A13}$$
$$p, p \Rightarrow q \vdash_{\mathbf{H}} q. \tag{MP}$$

LEMMA 5.1. *The following rules of inference are derived rules of* $\mathbf{H}$*:*

$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \wedge r) \Rightarrow (q \wedge s)$$
$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \vee r) \Rightarrow (q \vee s).$$

PROOF. See Appendix A. ■

LEMMA 5.2. *The deductive system* $\mathbf{H}$ *is regularly algebraisable with finite system of equivalence formulas* $\{p \Rightarrow q, q \Rightarrow p\}$*.*

PROOF. The proof of Raftery and van Alten [33, Proposition 2] shows that $\mathbf{H}$ satisfies Conditions (ALG1), (ALG4), and (ALG6). Condition (ALG3) holds for $\mathbf{H}$ trivially, while Condition (ALG2) follows from modus ponens. By the proof of [33, Proposition 2] again, $\mathbf{H}$ satisfies Condition (ALG5) with respect to the connectives $\Rightarrow$ and $*$. Further, Lemma 5.1 suffices to guarantee that $\mathbf{H}$ satisfies Condition (ALG5) with respect to the connectives $\wedge$ and $\vee$.

[12]The axioms and inference rules (A1)–(A13) and (MP) comprise a Hilbert-style presentation of $\mathbf{FL}_{ew}$ (see Lemma 5.4 below). For other Hilbert-style axiomatisations of $\mathbf{FL}_{ew}$, see Ono and Komori [28] and van Alten and Raftery [45]. Both these alternative axiomatisations enjoy the separation theorem. In contrast, the presentation of $\mathbf{FL}_{ew}$ given here lacks the separation theorem, but is convenient for applications.

Of course, Condition (ALG5) holds vacuously for $\mathbf{H}$ with respect to the connectives $\mathbf{0}$ and $\mathbf{1}$. Thus $\mathbf{H}$ is regularly algebraisable with finite system of equivalence formulas $\{p \Rightarrow q, q \Rightarrow p\}$. ■

By Condition (EQV1), $\mathsf{Alg\,Mod}^* \mathbf{H}$ satisfies an identity of the form $\varphi \approx \mathbf{1}$ for each axiom φ of the presentation of $\mathbf{H}$ given above. Denote any identity so obtained by $\varphi[\approx \mathbf{1}]$. By algebraisability and Conditions (AX1)–(AX3), $\mathsf{Alg\,Mod}^* \mathbf{H}$ is axiomatised by the identities (A1)$[\approx \mathbf{1}]$–(A13)$[\approx \mathbf{1}]$ together with the quasi-identities:

$$x \approx \mathbf{1} \ \text{ and } \ x \Rightarrow y \approx \mathbf{1} \ \text{ implies } \ y \approx \mathbf{1} \tag{5.1}$$

$$x \Rightarrow y \approx \mathbf{1} \ \text{ and } \ y \Rightarrow x \approx \mathbf{1} \ \text{ implies } \ x \approx y. \tag{5.2}$$

LEMMA 5.3. $\mathsf{Alg\,Mod}^* \mathbf{H}$ *is the variety of all* $\mathcal{FL}_{ew}$*-algebras.*

PROOF. Let $\mathbf{A} \in \mathsf{Alg\,Mod}^* \mathbf{H}$. From the proof of [33, Proposition 2], we have that the $\langle *, \Rightarrow, 1\rangle$-reducts of members of $\mathsf{Alg\,Mod}^* \mathbf{H}$ are pocrims. In particular, therefore, $\langle A; *, \Rightarrow, 1\rangle$ is a pocrim. Further, the identities (A6)$[\approx \mathbf{1}]$–(A8)$[\approx \mathbf{1}]$ and (A9)$[\approx \mathbf{1}]$–(A11)$[\approx \mathbf{1}]$ guarantee that for all $a, b \in A$, $a \wedge b$ and $a \vee b$ are the greatest lower bound and least upper bound of $\{a, b\}$ respectively with regards to the pocrim partial order $\sqsubseteq$.[13] Hence $\langle A; \wedge, \vee\rangle$ is a lattice whose lattice order $\leq$ is $\sqsubseteq$. By [41, Lemma 3.11], $\mathbf{A}$ is a commutative, integral, residuated lattice. The identity (A13)$[\approx \mathbf{1}]$ can now be seen to assert that $0 \leq a$ for all $a \in A$, whence $\mathbf{A} \in \mathcal{FL}_{ew}$. Hence $\mathsf{Alg\,Mod}^* \mathbf{H} \subseteq \mathcal{FL}_{ew}$.

Conversely, from the well-developed arithmetic of $\mathcal{FL}_{ew}$-algebras [8, 19] it readily follows that $\mathcal{FL}_{ew}$ satisfies the identities (A1)$[\approx \mathbf{1}]$–(A13)$[\approx \mathbf{1}]$ together with the quasi-identities (5.1)–(5.2). Hence $\mathcal{FL}_{ew} \subseteq \mathsf{Alg\,Mod}^* \mathbf{H}$. ■

LEMMA 5.4. $\mathbf{H}$ *is* $\mathbf{FL}_{ew}$.

PROOF. From Lemmas 5.2 and 5.3 we have that $\mathbf{H}$ is regularly algebraisable with equivalent variety semantics $\mathcal{FL}_{ew}$, while from Galatos and Ono [18, Theorems 3.3 and 3.4] we have that $\mathbf{FL}_{ew}$ is regularly algebraisable, also with equivalent variety semantics $\mathcal{FL}_{ew}$.[14] From Theorem 2.2(1) we conclude that $\mathbf{H} = \mathbf{S}^{\text{ASL}}\,\mathcal{FL}_{ew} = \mathbf{FL}_{ew}$ as desired. ■

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[13] For the definition of the pocrim partial order, see Part I of this series [41, Section 3].

[14] The results of [18, Theorem 3.3, Theorem 3.4] show only that $\mathbf{FL}_{ew}$ is algebraisable with equivalent variety semantics $\mathcal{FL}_{ew}$. However, it is easy to verify Condition (ALG6) holds for $\mathbf{FL}_{ew}$.

Let $\mathbf{NFL}_{ew}$ denote the axiomatic extension of $\mathbf{H}$ by the four axioms labelled (Double Negation), (Distributivity), (3-Potency), and (Nelson) of Section 1. Since any extension of a regularly algebraisable deductive system $\mathbf{S}$ is itself regularly algebraisable, from Lemma 5.2 we have that $\mathbf{NFL}_{ew}$ is regularly algebraisable. Moreover, from Lemma 5.4 and Condition (EQV1) we have that $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew}$ is the subvariety of $\mathcal{FL}_{ew}$ determined by the identities

$$\sim\sim x \Rightarrow x \approx \mathbf{1} \tag{5.3}$$

$$\big(x \wedge (y \vee z)\big) \Rightarrow \big((x \wedge y) \vee (x \wedge z)\big) \approx \mathbf{1} \tag{5.4}$$

$$\big(x \Rightarrow (x \Rightarrow (x \Rightarrow y))\big) \Rightarrow \big(x \Rightarrow (x \Rightarrow y)\big) \approx \mathbf{1} \tag{5.5}$$

$$\big((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))\big) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}. \tag{5.6}$$

In [23, p. 18] Kowalski and Ono essentially observe that a variety of $\mathcal{FL}_{ew}$-algebras satisfies (5.3) if and only if it satisfies (DN). By (5.3), therefore, $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew} \models$ (DN). Further, it is part of the folklore of lattice theory that a variety of lattices is distributive if and only if it satisfies the lattice inequality $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. From (5.4) it follows that $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew} \models$ (D7)–(D8). Additionally, it is well known from the theory of BCK-algebras that any class of BCK-algebras satisfying the BCK-identity $(x \Rightarrow^{n+1} y) \Rightarrow (x \Rightarrow^{n} y) \approx \mathbf{1}$ is $n+1$-potent.[15] From (5.5) we thus have that $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew} \models (\mathrm{E}_2)$. Summarising in the terminology of Part I [41]: $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew}$ is a variety of 3-potent, distributive, classical $\mathcal{FL}_{ew}$-algebras.

Lemma 5.5. *The variety* $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew}$ *satisfies the identity:*

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big(\sim y \Rightarrow (\sim y \Rightarrow \sim x)\big) \approx x \Rightarrow y. \tag{N}$$

Proof. See Appendix A. ∎

From Lemma 5.5 and the remarks directly preceding the lemma we have

Corollary 5.6. $\mathsf{Alg\,Mod}^*\,\mathbf{NFL}_{ew}$ *is the variety of Nelson* $\mathcal{FL}_{ew}$*-algebras.*

The main result of this paper, Theorem 1.1, now follows from directly from Corollary 5.6, Theorem 2.1, and Theorem 4.6.

By [41, Corollary 3.8], a Nelson algebra satisfies the identity $x \Rightarrow y \approx x \Rightarrow (x \Rightarrow y)$, where $\Rightarrow$ is defined as in $(\Rightarrow_{\mathrm{def}})$, if and only if it is term equivalent to a Boolean algebra. Thus $\mathcal{NFL}_{ew} \not\models x \Rightarrow y \approx x \Rightarrow (x \Rightarrow y)$. It follows that the deductive system $\mathbf{NFL}_{ew}$ is not contractive, *i.e.*, (c) is not a rule of $\mathbf{NFL}_{ew}$. From Theorem 3.1 we thus have

[15] For the definitions of the terms $x \Rightarrow^{n+1} y$ and $n+1$-potent, see [41, Section 3].

LEMMA 5.7.[16] $\mathbf{NFL}_{ew}$ *is a substructural logic over* $\mathbf{FL}_{ew}$.

The main result of this series of papers, Theorem 1.2, now follows directly from Theorem 1.1 and Lemma 5.7.

Added in proof. The results of this paper, together with results obtained recently by Busaniche and Cignoli in [11], imply $\mathbf{N}$ is definitionally equivalent to the extension $\mathbf{NFL}'_{ew}$ of the deductive system $\mathbf{H}$ by the axioms of (Double Negation), (3-potency), and the rule of inference

$$(p * p) \Rightarrow (q * q),\ (\sim p * \sim p) \Rightarrow (\sim q * \sim q) \vdash p \Rightarrow q.$$

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A. Appendix

In the following (machine-oriented) proof of Lemma 5.1, A, B, C and D denote arbitrary constants for which the hypothesis of the lemma holds and for which the corresponding conclusions necessarily follow. The justification $[i, j]$ indicates an application of modus ponens with major premise i and minor premise j. Steps 1–7 are axioms of $\mathbf{H}$; Steps 8 and 9 are the hypotheses of the lemma; and Steps 20 and 21 give the desired conclusions. Steps 20 and 21 of the proof are flagged with '*' for easy identification.

LEMMA 5.1. *The following rules of inference are derived rules of* $\mathbf{H}$*:*

$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \wedge r) \Rightarrow (q \wedge s)$$
$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \vee r) \Rightarrow (q \vee s).$$

[16] Lemma 5.7 continues to hold with respect to Galatos and Ono's conception of substructural logic over $\mathbf{FL}$. Hence the main result of this series of papers, Theorem 1.2, remains valid when formulated in the framework of [18, 19].

PROOF.

1. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$ [(A1)]
2. $(p \wedge q) \Rightarrow p$ [(A6)]
3. $(p \wedge q) \Rightarrow q$ [(A7)]
4. $(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$ [(A8)]
5. $p \Rightarrow (p \vee q)$ [(A9)]
6. $p \Rightarrow (q \vee p)$ [(A10)]
7. $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \vee r) \Rightarrow q))$ [(A11)]
8. $A \Rightarrow B$ [Assumption]
9. $C \Rightarrow D$ [Assumption]
10. $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow q)$ [1, 2]
11. $(p \Rightarrow q) \Rightarrow ((r \wedge p) \Rightarrow q)$ [1, 3]
12. $(B \Rightarrow p) \Rightarrow (A \Rightarrow p)$ [1, 8]
13. $(D \Rightarrow p) \Rightarrow (C \Rightarrow p)$ [1, 9]
14. $A \Rightarrow (B \vee p)$ [12, 5]
15. $(p \Rightarrow (B \vee q)) \Rightarrow ((A \vee p) \Rightarrow (B \vee q))$ [7, 14]
16. $C \Rightarrow (p \vee D)$ [13, 6]
17. $(A \wedge p) \Rightarrow B$ [10, 8]
18. $((A \wedge p) \Rightarrow q) \Rightarrow ((A \wedge p) \Rightarrow (B \wedge q))$ [4, 17]
19. $(p \wedge C) \Rightarrow D$ [11, 9]

*20. $(A \vee C) \Rightarrow (B \vee D)$ [15, 16]

*21. $(A \wedge C) \Rightarrow (B \wedge D)$ [18, 19]

■

In the (machine-oriented) proof of Lemma 5.5 below, the justification $[i \rightarrow j]$ indicates paramodulation from i into j; that is, unifying the left-hand side of i with a subterm of j, instantiating j with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of i. The labels (D3), (M1), etc., in Steps 1–2, 4, and 6–10 indicate identities established in Part I [41].

LEMMA 5.5. *The variety* $\mathsf{Alg\,Mod}^* \mathbf{NFL}_{ew}$ *satisfies the identity:*

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim}\, y \Rightarrow ({\sim}\, y \Rightarrow {\sim}\, x)\big) \approx x \Rightarrow y. \tag{N}$$

PROOF.

1. $x \vee y \approx y \vee x$ [(D3)]
2. $x * \mathbf{1} \approx x$ [(M1)]
3. ${\sim}\, x := x \Rightarrow \mathbf{0}$ [(${\sim}_{\mathrm{def}}$)]
4. ${\sim}\,{\sim}\, x \approx x$ [(DN)]
5. $((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim}\, y \Rightarrow ({\sim}\, y \Rightarrow {\sim}\, x))) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}$ [(5.6)]

6. $x \Rightarrow (y \Rightarrow x) \approx \mathbf{1}$ [(3.17)]
7. $x \Rightarrow (y \Rightarrow z) \approx y \Rightarrow (x \Rightarrow z)$ [(3.18)]
8. $(x \Rightarrow y) \wedge (x \Rightarrow z) \approx x \Rightarrow (y \wedge z)$ [(4.2)]
9. $(x * (x \Rightarrow y)) \vee y \approx y$ [(4.3)]
10. $(x \Rightarrow y) \wedge (z \Rightarrow y) \approx (x \vee z) \Rightarrow y$ [(4.4)]
11. $x \Rightarrow (y \Rightarrow \mathbf{0}) \approx y \Rightarrow \sim x$ [$3 \rightarrow 7$]
12. $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow y) \approx \mathbf{1}$ [$7 \rightarrow 5$]
13. $x \vee (y * (y \Rightarrow x)) \approx x$ [$9 \rightarrow 1$]
14. $(x \Rightarrow y) \vee (y * \mathbf{1}) \approx x \Rightarrow y$ [$6 \rightarrow 13$]
15. $(x \Rightarrow y) \vee (z * (x \Rightarrow (z \Rightarrow y))) \approx x \Rightarrow y$ [$7 \rightarrow 13$]
16. $x \Rightarrow \sim y \approx y \Rightarrow \sim x$ [$3 \rightarrow 11$]
17. $\sim x \Rightarrow \sim y \approx y \Rightarrow x$ [$4 \rightarrow 16$]
18. $(x \Rightarrow y) \vee y \approx x \Rightarrow y$ [$2 \rightarrow 14$]
19. $x \vee (y \Rightarrow x) \approx y \Rightarrow x$ [$18 \rightarrow 1$]
20. $(x \Rightarrow y) \vee (x \Rightarrow (z \Rightarrow y)) \approx z \Rightarrow (x \Rightarrow y)$ [$7 \rightarrow 19$]
21. $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (x \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$ [$17 \rightarrow 12$]
22. $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (x \Rightarrow (\sim y \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$ [$7 \rightarrow 21$]
23. $x \Rightarrow ((x \Rightarrow ((x \Rightarrow y) \wedge (\sim y \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$ [$8 \rightarrow 22$]
24. $x \Rightarrow ((x \Rightarrow ((x \vee \sim y) \Rightarrow y)) \Rightarrow y) \approx \mathbf{1}$ [$10 \rightarrow 23$]
25. $(x \Rightarrow y) \vee ((x \Rightarrow ((x \vee \sim y) \Rightarrow y)) * \mathbf{1}) \approx x \Rightarrow y$ [$24 \rightarrow 15$]
26. $(x \Rightarrow y) \vee (x \Rightarrow ((x \vee \sim y) \Rightarrow y)) \approx x \Rightarrow y$ [$2 \rightarrow 25$]
27. $(x \vee \sim y) \Rightarrow (x \Rightarrow y) \approx x \Rightarrow y$ [$26 \rightarrow 20$]
28. $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (x \Rightarrow y)) \approx x \Rightarrow y$ [$27 \rightarrow 10$]
29. $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y$ [$17 \rightarrow 28$]

■

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Matthew Spinks
Mathematical Institute
University of Bern
CH-3012 Bern, Switzerland
`mspinksau@yahoo.com.au`

Robert Veroff
Department of Computer Science
University of New Mexico
Albuquerque NM 87131, U.S.A.
`veroff@cs.unm.edu`